

# Reflected entropy in holography and beyond

Based on: arXiv: 2107.00009

The Markov gap for geometric reflected entropy  
H, Parrikar, Sorce

## Reflected entropy: why and what?

- Ryu-Takayangi formula



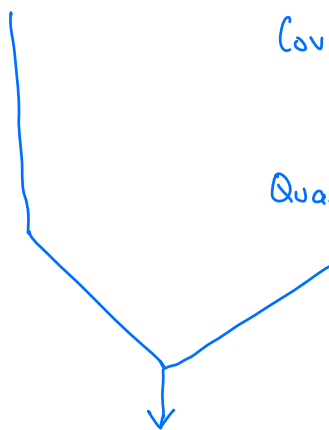
Entanglement wedge  
reconstruction



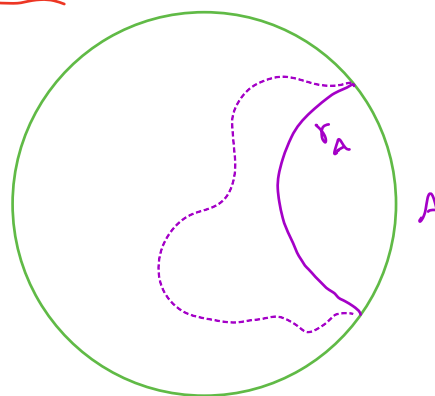
Covariant HRT formula



Quantum extremal surface



Calculation of entropy  
of Hawking radiation



- Big payoff to properly understanding quantum correlations in holographic theories

- Which quantities have simple / compelling bulk interpretations?
- Can they say anything about multipartite entanglement?

### Canonical purifications

- Given density operator  $\rho \in \mathcal{H} \otimes \mathcal{H}^*$
- Treat  $\mathcal{H} \otimes \mathcal{H}^*$  as Hilbert space with

$$\langle\langle \rho | \sigma \rangle\rangle = \text{tr}(\rho^\dagger \sigma)$$

- Then  $|\sqrt{\rho}\rangle\rangle$  is a normalized state:

$$\langle\langle \sqrt{\rho} | \sqrt{\rho} \rangle\rangle = \text{tr}(\sqrt{\rho}^\dagger \sqrt{\rho}) = \text{tr} \rho = 1$$

- Purification:  $\text{tr}_{\mathcal{H}_L^* \otimes \mathcal{H}_R} |\sqrt{\rho}\rangle\rangle_{\mathcal{H}_L \otimes \mathcal{H}_L^*} \langle\langle \rho |_{\mathcal{H}_R^* \otimes \mathcal{H}_R} = \rho$

### Reflected entropy [Dutta, Faulkner 19]

- Measure of correlation<sup>(\*)</sup> in  $\mathcal{S}_{AB}$

$$S_R(A:B)_\rho = S(AA^*)_{\rho\rho}$$

$\uparrow$  entanglement entropy of  $\rho\rho$   
 $\uparrow$  Canonical purification  
 can be mixed

## Simple examples

- $|\psi\rangle_{AB}$  is pure

$$\bullet |\sqrt{\rho}\rangle\rangle_{AA^*BB^*} = |\psi\rangle_{AB} \langle\psi|_{A^*B^*}$$

$$\bullet S_R(A:B) = S(AA^*)_{\sqrt{\rho}} = 2S(A)_\psi$$

$\uparrow$   
 $!!$

- $\rho_{AB} = \rho_A \otimes \tau_B$  is product

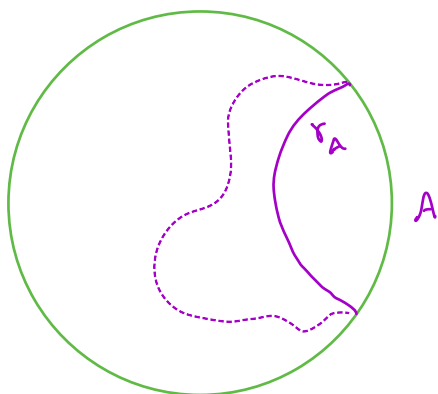
$$\bullet |\sqrt{\rho}\rangle\rangle_{AA^*BB^*} = |\sqrt{\rho}\rangle\rangle_{AA^*} \otimes |\sqrt{\tau}\rangle\rangle_{BB^*}$$

$$\bullet S_R(A:B) = S(AA^*)_{\sqrt{\rho}} = 0$$

$\nwarrow$  pure state on  $AA^*$ !

# Holographic interpretation [Dutta, Faulkner '19]

Recall Ryu-Takayanagi:



$$S(A) = \frac{1}{4G_N} \min_{\gamma \sim A} \text{Area}(\gamma)$$

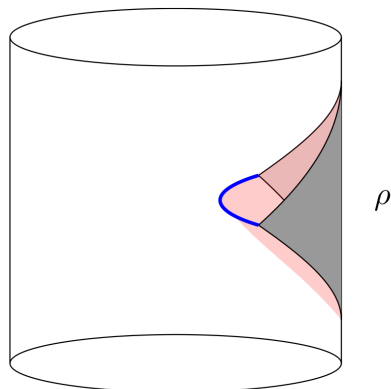
$\gamma \sim A$ :  $\gamma$  homologous to  $A$

ie:  $\gamma \cup A = \partial(\text{Something})$

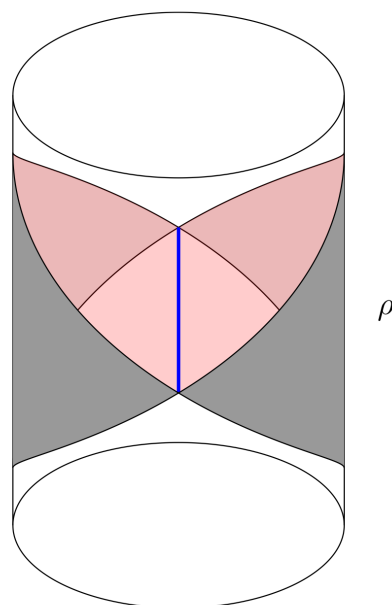
Entanglement wedge:

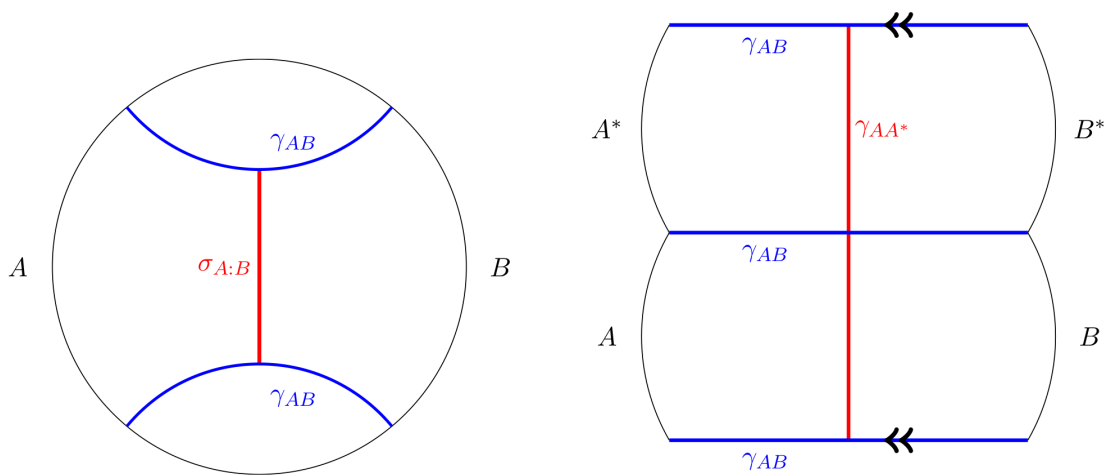
$$\gamma_A \cup A = \partial[W(A)]$$

- Glue entanglement wedge  $W(p)$  to its CPT conjugate along (quantum) extremal surface forming the spatial boundary of  $W(p)$ .



CPT( $\rho$ )





- Canonical purification of  $\rho_{AB}$  contains a wormhole

$$S(AA^*)_{\rho_P} = \frac{1}{4G_N} \text{len } \gamma_{AA^*}$$

$$= \frac{1}{4G_N} 2 \text{len } \underbrace{\sigma_{A:B}}$$

↑ minimal cross-section  
of  $W(AB)$

- Other candidates for cross-section

- Entanglement of purification [Takayanagi, Umemoto 18]
- Logarithmic negativity [Nguyen et al. 18]
- Balanced partial entanglement [Wen 19]

## Markov gap & multipartite entanglement

$$\Delta := S_R(A:B) - I(A:B)$$

$$= S(AA^*) - [S(A) + S(B) - S(AB)]$$

$$= S(AA^*) - S(A) - S(AA^*B^*) + S(AB)$$

↑ globally pure on  $AA^*BB^*$

$$= I(A^*:B|A) \geq 0$$

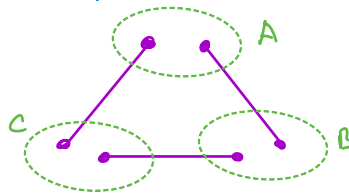
↑ Conditional mutual information

Note [Akers-Rath 19]:  $\Delta$  detects multipartite entanglement

For pure  $|\psi\rangle_{ABC}$ ,

$$\Delta = 0 \Leftrightarrow$$

only bipartite entanglement

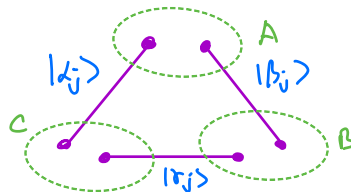


[Zou et al. 21]

$$\Delta = 0 \Leftrightarrow$$

Sum of triangle states

$$\sum_j \sqrt{p_j}$$

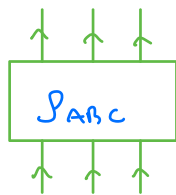


## Markov gap in holographic states

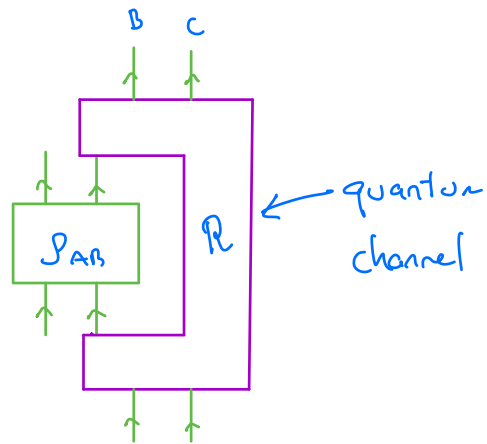
- Intuition from universal recovery maps

[Fawzi, Renner '15]

- Try to make



by

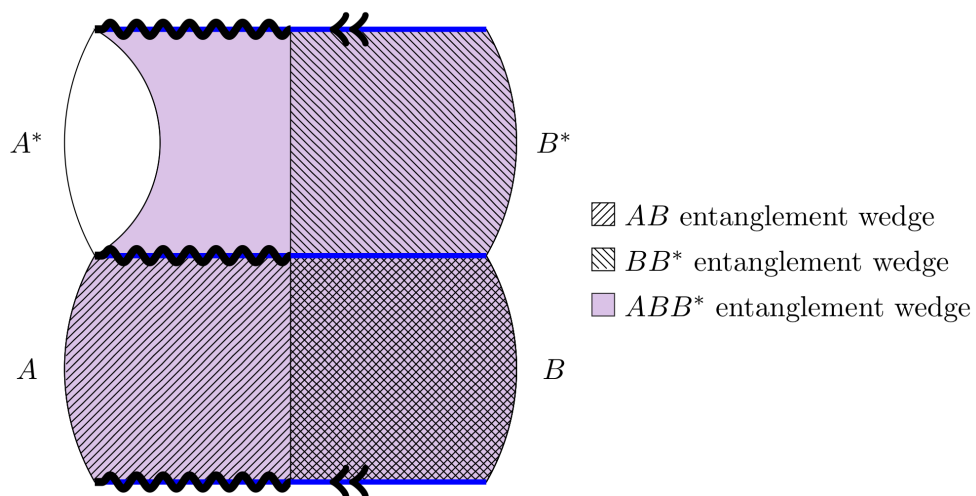


- $I(A:C|B)$  is obstruction to making  $P_{ABC}$  this way

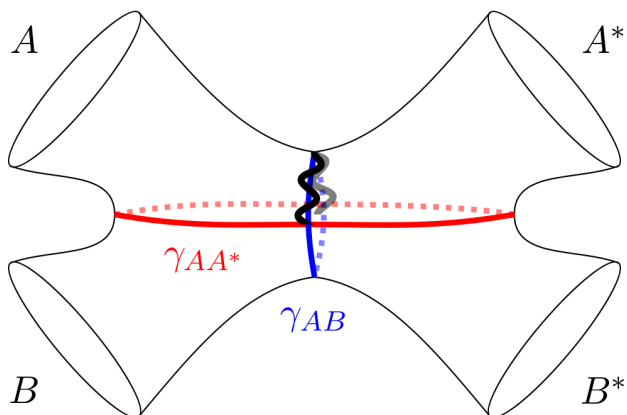
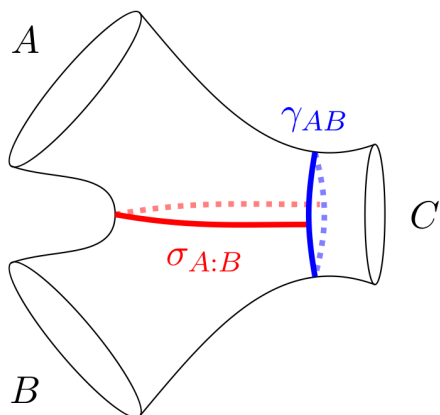
$$\max_{R_{B \rightarrow BC}} F(P_{ABC}, R_{B \rightarrow BC}(P_{AB})) \geq e^{-I(A:C|B)}$$

- Recall  $\Delta := S_R(A:B) - I(A:B)$   
 $= I(A: B^* | B)$

- Consider trying to make  $\mathcal{I}_{AB|B^*}$  from  $\mathcal{I}_{AB}$  by acting on  $B$  alone



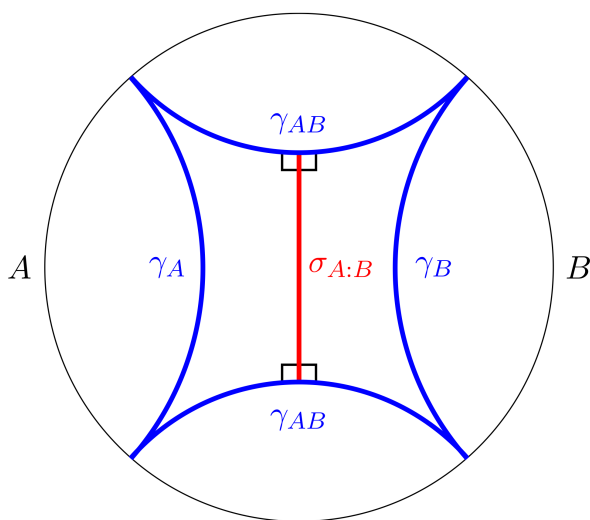
- Cannot expect to properly reproduce bulk correlations across jagged  $\sim$  (and probably fail in other ways too!)



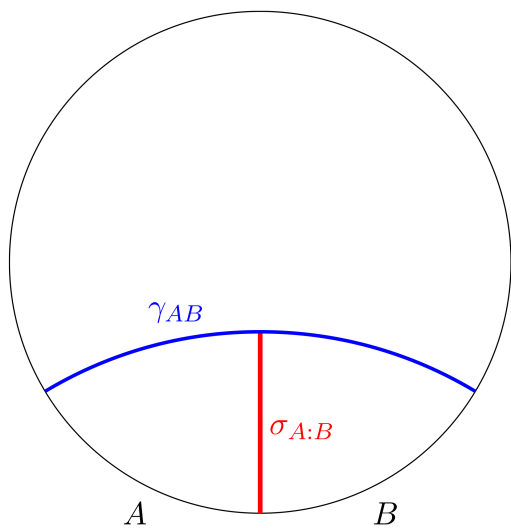
- In this case,  $\sim$  Surface is compact and can be made arbitrarily small
- Nonetheless, will find irreducible contribution to  $\Delta$  from corner where  $\sim$  intersects entanglement wedge cross section  $\delta_{A:B}$
- In pure gravity  $\text{AdS}_3 / \text{CFT}_2$

$$\Delta \geq \frac{l_{\text{AdS}} \log(2)}{2 G_N} \times (\# \text{ cross section boundaries}) + o(1/G_N)$$

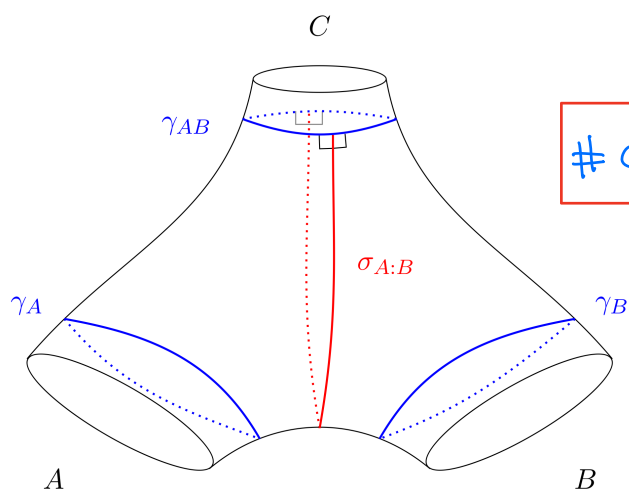
Examples



$$\# \text{ cross section boundaries} = 2$$

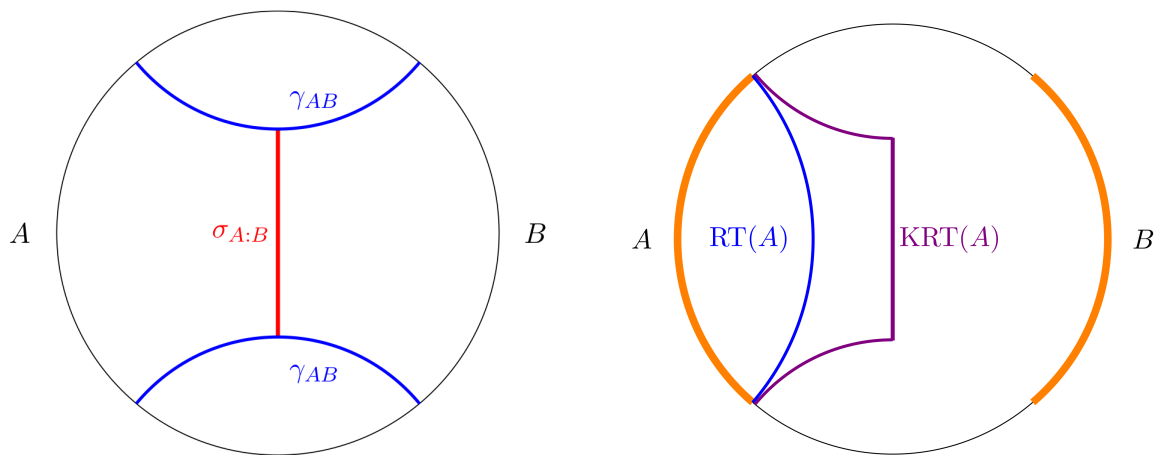


# cross section boundaries = 1



# cross section boundaries = 2

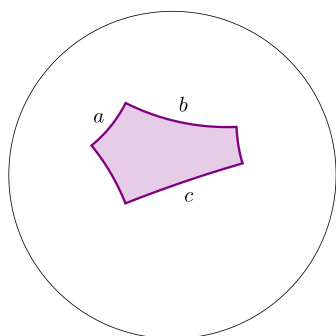
Proof idea



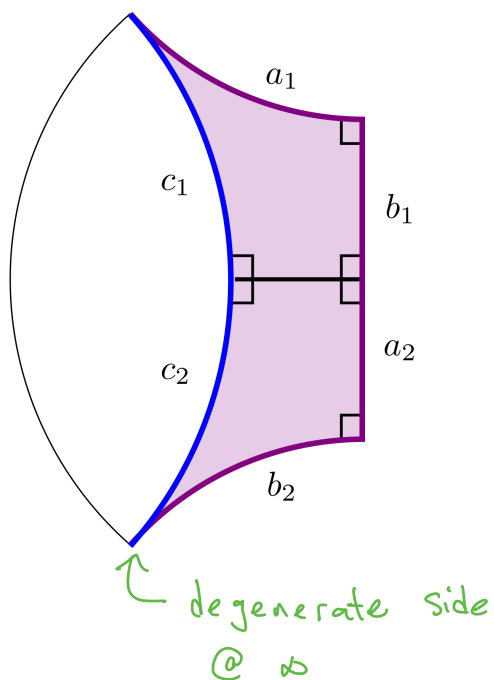
$$\Delta = S_R(A:B) - I(A:B)$$

$$= \frac{\text{area}[KRT(A)] - \text{area}[RT(A)]}{4 G_N} + \frac{\text{area}[KRT(B)] - \text{area}[RT(B)]}{4 G_N}$$

Right-angled hyperbolic pentagons



$$a + b - c \geq \log 2$$



- $a_j + b_j - c_j \geq \log 2$
- $\frac{\text{area}[KRT(A)] - \text{area}[RT(A)]}{4G_n} \geq 2 \log 2$
- $\Delta = S_R - I \geq \frac{\log 2}{G_n}$

### General case:

- Spatial slice is quotient of hyperbolic plane
- Work in covering space
- Each KRT is homotopic to a smooth geodesic
  - Region between them tiled by right-angled hyperbolic pentagons
  - Length inequality applies
  - RT area  $\leq$  geodesic area

## Take-home message

- Multipartite entanglement, witnessed by  $\Delta > 0$ , associated to codimension-3 structures in space-time

## What next?

- Higher dimensions? Matter?
- States without  $t \rightarrow -t$  symmetry?
- Is  $S_R(A:B)$  a measure of correlation?

$$S_R(A:BC) \stackrel{?}{\geq} S_R(A:B)$$

- Easy to show holographically
- Would imply (depending on stds of rigor)

Holographic entanglement of purification conjecture

- $E_P(A:B) = \frac{1}{2} S_R(A:B)$  for holographic states

Also: formula for distillable entanglement of holographic states